Boolean Algebra

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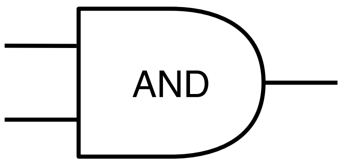
Core Boolean Operators

AND

$\wedge$

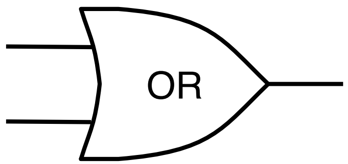
A	B	A AND B
0	0	0
0	1	0
1	0	0
1	1	1



OR

$\vee$

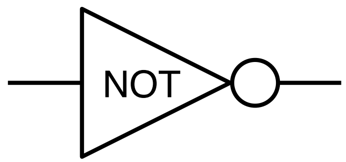
A	B	A OR B
0	0	0
0	1	1
1	0	1
1	1	1



NOT

$\neg$

A	NOT A
0	1
1	0



from Boolean expressions to circuits

Truth Table → Boolean Expression

- Truth table

A	B	OUT
0	0	1
0	1	0
1	0	0
1	1	0

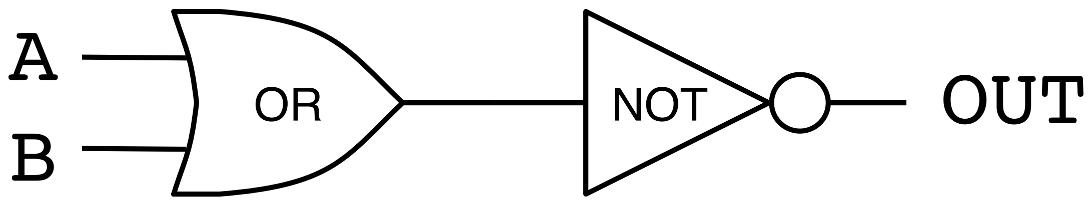
- Operation: NOT (A OR B)

  (also called NOR)

Boolean Expression → Circuit

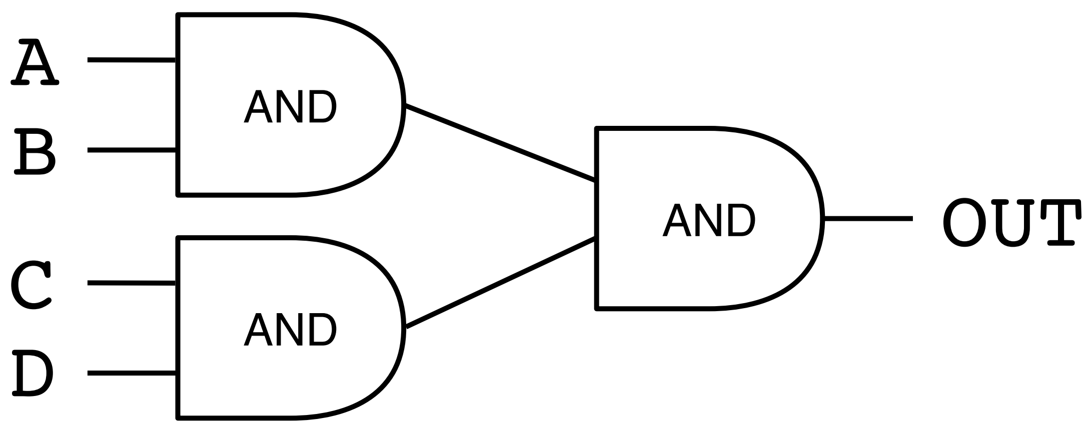
- Operation: NOT (A OR B)

- Circuit:



4-Bit AND

- 4 inputs (A, B, C, D), output 1 iff all inputs are 1
- Operation: (A AND B) AND (C AND D)
- Circuit:



1-Bit Selector

- Truth table

A	OUT1	OUT2
0	1	0
1	0	1

- Operation: OUT1 = NOT A

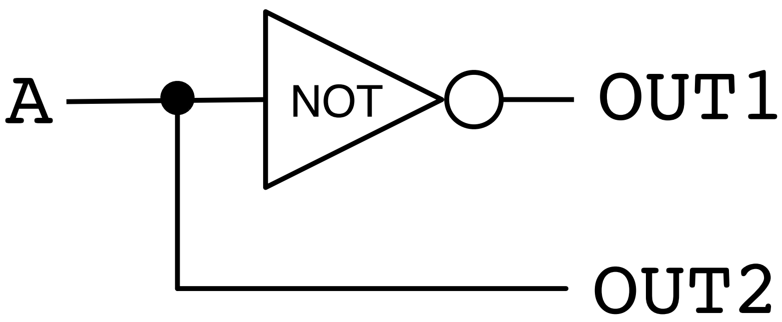
  OUT2 = A

1-Bit Selector

- Operation: OUT1 = NOT A

  OUT2 = A

- Circuit:



A Complicated Example

- Truth table

A	B	C	OUT
0	0	0	0
0	1	0	1
1	0	0	0
1	1	0	0
0	0	1	1
0	1	1	1
1	0	1	0
1	1	1	0

- Operation: Need a better way of doing this instead of relying on intuition

disjunctive normal form

DNF: Setup

A	B	C	OUT	Expression
0	0	0	0	
0	1	0	1	
1	0	0	0	
1	1	0	0	
0	0	1	1	
0	1	1	1	
1	0	1	0	
1	1	1	0	

Goal: find expression for each row that yields 1

DNF: One Row

A	B	C	OUT	Expression
0	0	0	0	
0	1	0	1	(NOT A) AND B AND (NOT C)
1	0	0	0	
1	1	0	0	
0	0	1	1	
0	1	1	1	
1	0	1	0	
1	1	1	0	

Expression is 1 only for this row, 0 for all others

DNF: All Rows

A	B	C	OUT	Expression
0	0	0	0	
0	1	0	1	(NOT A) AND B AND (NOT C)
1	0	0	0	
1	1	0	0	
0	0	1	1	(NOT A) AND (NOT B) AND C
0	1	1	1	(NOT A) AND B AND C
1	0	1	0	
1	1	1	0	

DNF: Complete Operation

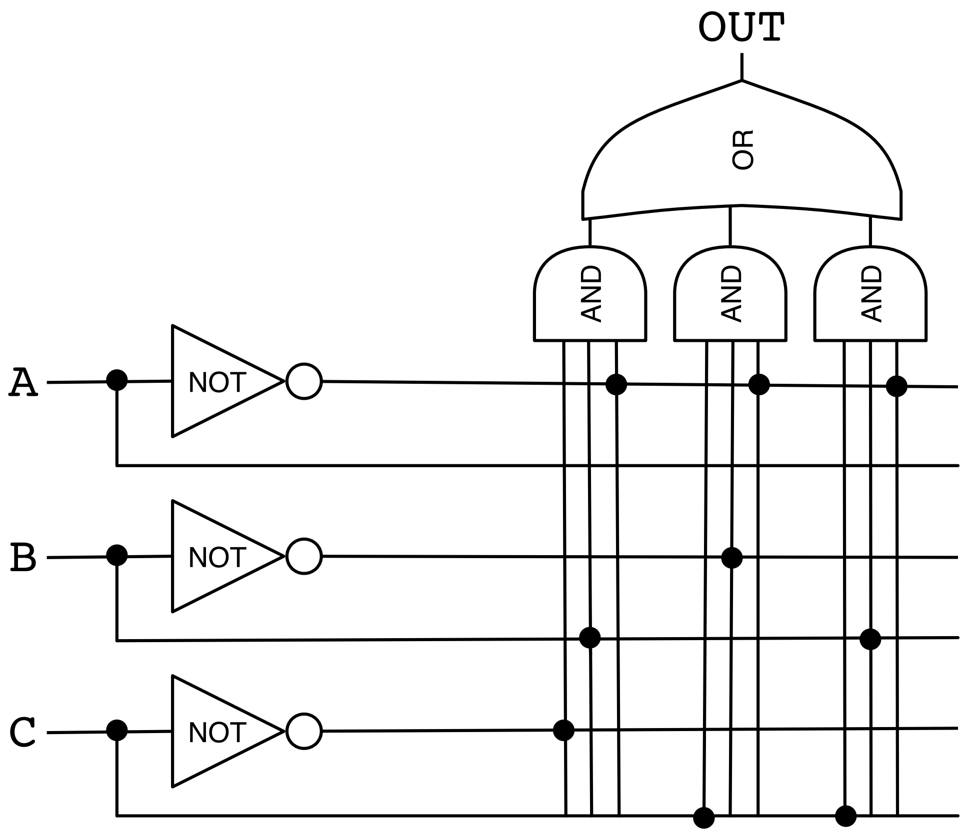
A	B	C	OUT	Expression
0	0	0	0	
0	1	0	1	(NOT A) AND B AND (NOT C)
1	0	0	0	
1	1	0	0	
0	0	1	1	(NOT A) AND (NOT B) AND C
0	1	1	1	(NOT A) AND B AND C
1	0	1	0	
1	1	1	0	

Putting it all together: ((NOT A) AND B AND (NOT C)) OR
((NOT A) AND (NOT B) AND C) OR
((NOT A) AND B AND C)

DNF: Circuit

- Operation: ((NOT A) AND B AND (NOT C)) OR
  ((NOT A) AND (NOT B) AND C) OR
  ((NOT A) AND B AND C)

- Circuit:



conjunctive normal form

DNF

A	B	C	OUT	Expression
0	0	0	0	
0	1	0	1	(NOT A) AND B AND (NOT C)
1	0	0	0	
1	1	0	0	
0	0	1	1	(NOT A) AND (NOT B) AND C
0	1	1	1	(NOT A) AND B AND C
1	0	1	0	
1	1	1	0	

Putting it all together: ((NOT A) AND B AND (NOT C)) OR
((NOT A) AND (NOT B) AND C) OR
((NOT A) AND B AND C)

CNF: One Row

A	B	C	OUT	Expression
0	0	0	0	NOT ((NOT A) AND (NOT B) AND (NOT C))
0	1	0	1	
1	0	0	0	
1	1	0	0	
0	0	1	1	
0	1	1	1	
1	0	1	0	
1	1	1	0	

Expression is 0 only for this row, 1 for all others

CNF: All Rows

A	B	C	OUT	Expression
0	0	0	0	NOT ((NOT A) AND (NOT B) AND (NOT C))
0	1	0	1	
1	0	0	0	NOT (A AND (NOT B) AND (NOT C))
1	1	0	0	NOT (A AND B AND (NOT C))
0	0	1	1	
0	1	1	1	
1	0	1	0	NOT (A AND (NOT B) AND C)
1	1	1	0	NOT (A AND B AND C)

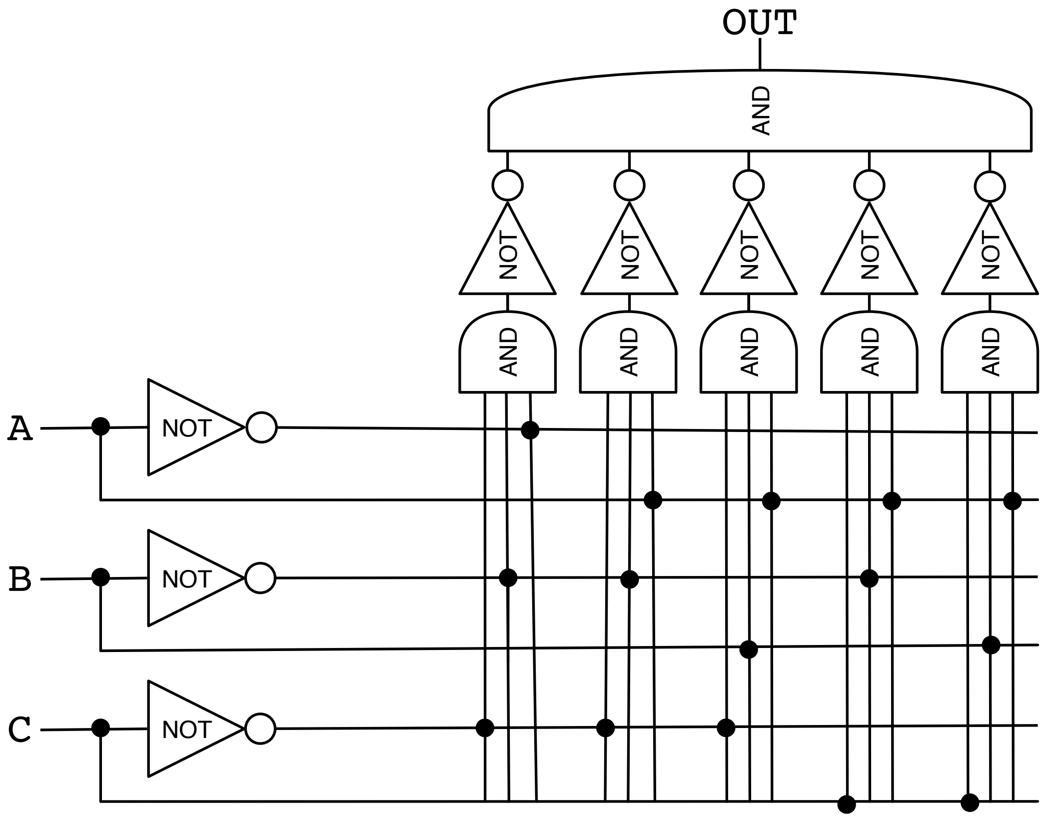
CNF: Complete Operation

A	B	C	OUT	Expression
0	0	0	0	NOT ((NOT A) AND (NOT B) AND (NOT C))
0	1	0	1	
1	0	0	0	NOT (A AND (NOT B) AND (NOT C))
1	1	0	0	NOT (A AND B AND (NOT C))
0	0	1	1	
0	1	1	1	
1	0	1	0	NOT (A AND (NOT B) AND C)
1	1	1	0	NOT (A AND B AND C)

Putting it all together: (NOT ((NOT A) AND (NOT B) AND (NOT C))) AND
(NOT (A AND (NOT B) AND (NOT C))) AND
(NOT (A AND B AND (NOT C))) AND
(NOT (A AND (NOT B) AND C)) AND
(NOT (A AND B AND C))

CNF: Circuit

- Operation: (NOT ((NOT A) AND (NOT B) AND (NOT C))) AND
  (NOT (A AND (NOT B) AND (NOT C))) AND
  (NOT (A AND B AND (NOT C))) AND
  (NOT (A AND (NOT B) AND C)) AND
  (NOT (A AND B AND C))
- Circuit:



universal gates

Universality of NAND

- Truth table:

A	B	A nand B
0	0	1
0	1	1
1	0	1
1	1	0

- NOT: A nand A

- AND: (A nand B) nand (A nand B)

- OR: (A nand A) nand (B nand B)

Universality of NOR

- Truth table:

A	B	A NOR B
0	0	1
0	1	0
1	0	0
1	1	0

- NOT: A NOR A
- AND: (A NOR A) NOR (B NOR B)
- OR: (A NOR B) NOR (A NOR B)

Numbers


There are only 10 kinds of people.
Those who understand binary
and those who don't.

Roman Numerals

- Basic units

I	V	X	L	C	D	M
1	5	10	50	100	500	1000

- Additive combination of units

II	III	VI	XVI	XXXIII	MDCLXVI	MMXVI
2	3	6	16	33	1666	2016

- Subtractive combination of units

IV	IX	XL	XC	CD	CM	MCMLXXI
4	9	40	90	400	900	1971

Arabic Numerals

- Developed in India and Arabic world during the European Dark Age
- Decisive step: invention of zero by Brahmagupta in AD 628
- Basic units

0 1 2 3 4 5 6 7 8 9

- Positional system

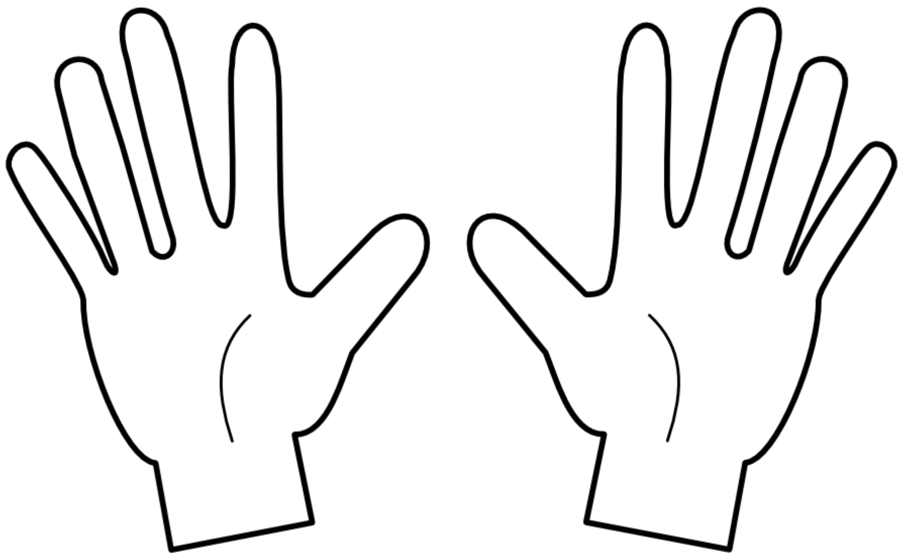
1 10 100 1000 10000 100000 1000000

Why Base 10?

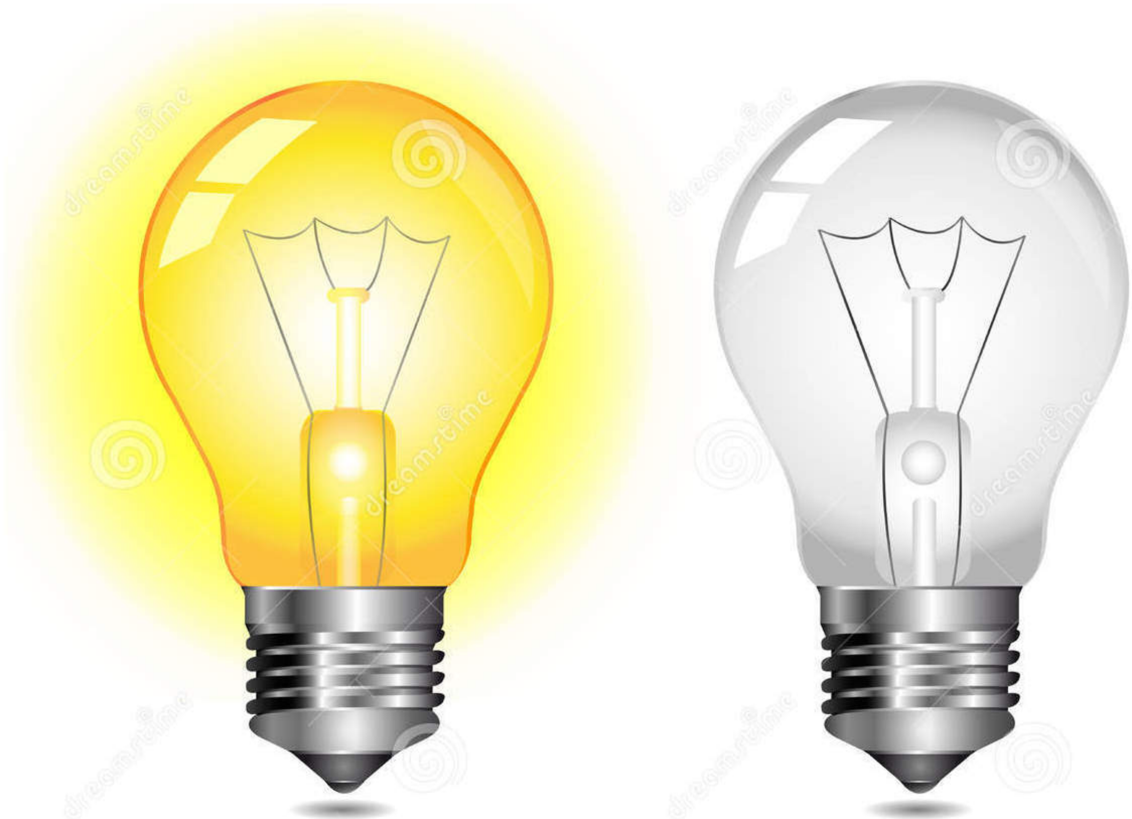
dig·it
/ˈdijit/

noun

1. any of the numerals from 0 to 9, especially when forming part of a number.
   synonyms: numeral, number, figure, integer
   "the door code has ten digits"
2. a finger (including the thumb) or toe.
   synonyms: finger, thumb, toe; extremity
   "we wanted to warm our frozen digits"



Base 2



Base 2

- Decoding binary numbers

Binary number	1	1	0	1	0	1	0	1	
Position	7	6	5	4	3	2	1	0	
Value	2^7	2^6	0	2^4	0	2^2	0	2^0	
	128	64	0	16	0	4	0	1	= 213

Base 8

- Numbers like 11010101 are very hard to read

⇒ Octal numbers

Binary number	1 1	0 1 0	1 0 1	
	-----	--------	--------	
Octal number	3	2	5	
Position	2	1	0	
Value	3×8^2	2×8^1	5×8^0	
	192	16	5	= 213

- ... but grouping **three** binary digits is a bit odd

Base 16

- Grouping 4 binary digits $\rightarrow$ base $2^4 = 16$
- "Hexadecimal" (hex = Greek for six, decimus = Latin for tenth)
- Need characters for 10-15: use letters a-f

Binary number	1	1	0	1	0	1	0	1	
	----------				----------				
Hexadecimal number	d				5				
Position	1				0				
Value	13×16^1				5×16^0				
	208				5				= 213

Examples

Decimal	Binary	Octal	Hexademical
0			
1			
2			
3			
8			
15			
16			
20			
23			
24			
30			
50			
100			
255			
256			

Examples

Decimal	Binary	Octal	Hexademical
0	0	0	0
1	1	1	1
2	10	2	2
3	11	3	3
8	1000	10	8
15	1111	17	f
16	10000	20	10
20	10100	24	14
23	10111	27	17
24	11000	30	18
30	11110	36	1e
50	110010	62	32
100	1100100	144	64
255	11111111	377	ff
256	100000000	400	100

adding binary numbers

Binary Addition

- Adding binary numbers - just like decimal numbers

A	0	1	0	1	0	1	0	1
B	0	0	0	1	1	1	0	0
Carry								
A+B								

- Problem setup

Binary Addition

- Adding binary numbers - just like decimal numbers

A	0	1	0	1	0	1	0	1
B	0	0	0	1	1	1	0	0
Carry							-	
A+B								1

- Adding the last two digits: 1 + 0 = 1

Binary Addition

- Adding binary numbers - just like decimal numbers

A	0	1	0	1	0	1	0	1
B	0	0	0	1	1	1	0	0
Carry						-	-	
A+B							0	1

- Adding the next two digits: 0 + 0 = 0

Binary Addition

- Adding binary numbers - just like decimal numbers

A	0	1	0	1	0	1	0	1
B	0	0	0	1	1	1	0	0
Carry					1	-	-	
A+B						0	0	1

- Adding the next two digits: 1 + 1 = 0, carry 1

Binary Addition

- Adding binary numbers - just like decimal numbers

A	0	1	0	1	0	1	0	1
B	0	0	0	1	1	1	0	0
Carry				1	1	-	-	
A+B					0	0	0	1

- Adding the next two digits, plus carry : 0 + 1 + 1 = 0, carry 1

Binary Addition

- Adding binary numbers - just like decimal numbers

A	0	1	0	1	0	1	0	1
B	0	0	0	1	1	1	0	0
Carry			1	1	1	-	-	
A+B				1	0	0	0	1

- Adding the next two digits, plus carry : 1 + 1 + 1 = 0, carry 1

Binary Addition

- Adding binary numbers - just like decimal numbers

A	0	1	0	1	0	1	0	1
B	0	0	0	1	1	1	0	0
Carry	-	-	1	1	1	-	-	
A+B	0	1	1	1	0	0	0	1

- And so on...

negative numbers

Positive Numbers

Bits			Unsigned			
0	0	0	0			
0	0	1	1			
0	1	0	2			
0	1	1	3			
1	0	0	4			
1	0	1	5			
1	1	0	6			
1	1	1	7			

- Encoding for unsigned binary numbers

One Bit for Sign

Bits			Unsigned	Sign + Magnitude		
0	0	0	0	+0		
0	0	1	1	+1		
0	1	0	2	+2		
0	1	1	3	+3		
1	0	0	4	-0		
1	0	1	5	-1		
1	1	0	6	-2		
1	1	1	7	-3		

- Use the first bit to encode sign: 0 = positive, 1 = negative
- How can we do addition with this?

One's Complement

Bits			Unsigned	Sign + Magnitude	One's Complement	
0	0	0	0	+0	+0	
0	0	1	1	+1	+1	
0	1	0	2	+2	+2	
0	1	1	3	+3	+3	
1	0	0	4	-0	-3	
1	0	1	5	-1	-2	
1	1	0	6	-2	-1	
1	1	1	7	-3	-0	

- Negative number: flip all bits
- Some waste: two zeros (+0=000 and -0=111)

Two's Complement

Bits			Unsigned	Sign + Magnitude	One's Complement	Two's Complement
0	0	0	0	+0	+0	+0
0	0	1	1	+1	+1	+1
0	1	0	2	+2	+2	+2
0	1	1	3	+3	+3	+3
1	0	0	4	-0	-3	-4
1	0	1	5	-1	-2	-3
1	1	0	6	-2	-1	-2
1	1	1	7	-3	-0	-1

- Negative number: flip all bits, add 001
- Addition works as before: -1 + -1 = 111 + 111 = 1110 = -2
  +2 + -1 = 010 + 111 = 1001 = +1