# Floating Point Numbers

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# Numbers



- So far, we only dealt with integers
- But there are other types of numbers
- Rational numbers (from ratio $\simeq$ fraction)
  - 3/4 = 0.75
  - 10/3 = 3.33333333....
- Real numbers
  - $\pi$ = 3.14159265...
  - e = 2.71828182...

# Very Large Numbers



- Distance of sun and earth

$$150,000,000,000 \text{ meters}$$

- Scientific notation

$$1.5 \times 10^{11} \text{ meters}$$

- Another example: number of atoms in 12 gram of carbon-12 (1 mol)

$$6.022140857 \times 10^{23}$$

# Binary Numbers in Scientific Notation



- Example binary number ($\pi$ again)

$$11.0010010001$$

- Scientific notation

$$1.10010010001 \times 2^1$$

- General form

$$1.x \times 2^y$$

# Representation

- IEEE 754 floating point standard

- Uses 4 bytes

| 31 | 30 | 29 | 28 | 27 | 26 | 25 | 24 | 23 | 22 | 21 | 20 | ... | 2 | 1 | 0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| s | exponent | | | | | | | | fraction | | | | | | |
| 1 bit | 8 bits | | | | | | | | 23 bits | | | | | | |

- Exponent is offset with a bias of 127

  e.g. $2^{-6} \rightarrow$ exponent = -6 + 127 = 121

# Conversion into Binary



- $\pi = 3.14159265$
- Number before period: $3_{10} = 11_2$
- Conversion of fraction .14159265

| Digit | Calculation | Digit | Calculation |
|---|---|---|---|
| | $0.14159265 \times 2 \downarrow$ | 1 | $0.9817472 \times 2 \downarrow$ |
| 0 | $0.2831853 \times 2 \downarrow$ | 1 | $0.9634944 \times 2 \downarrow$ |
| 0 | $0.5663706 \times 2 \downarrow$ | 1 | $0.9269888 \times 2 \downarrow$ |
| 1 | $0.1327412 \times 2 \downarrow$ | 1 | $0.8539776 \times 2 \downarrow$ |
| 0 | $0.2654824 \times 2 \downarrow$ | 1 | $0.7079552 \times 2 \downarrow$ |
| 0 | $0.5309648 \times 2 \downarrow$ | 1 | $0.4159104 \times 2 \downarrow$ |
| 1 | $0.0619296 \times 2 \downarrow$ | 0 | $0.8318208 \times 2 \downarrow$ |
| 0 | $0.1238592 \times 2 \downarrow$ | 1 | $0.6636416 \times 2 \downarrow$ |
| 0 | $0.2477184 \times 2 \downarrow$ | 1 | $0.3272832 \times 2 \downarrow$ |
| 0 | $0.4954368 \times 2 \downarrow$ | 0 | $0.6545664 \times 2 \downarrow$ |
| 0 | $0.9908736 \times 2 \rightarrow$ | 1 | $0.3091328 \times 2$ |

- Binary: 11.001001000011111101101

# Encoding into Representation



- $\pi$

$$1.10010010000111111101101 \times 2^1$$

- Encoding

| Sign | Exponent | Fraction |
|---|---|---|
| 0 | 10000000 | 10010010000111111101101 |

- Note: leading 1 in fraction is omitted

# Special Cases

- Zero
- Infinity (1/0)
- Negative infinity (-1/0)
- Not a number (0/0 or $\infty - \infty$)

# Encoding

| Exponent | Fraction | Object |
|---|---|---|
| 0 | 0 | zero |
| 0 | >0 | denormalized number |
| 1-254 | anything | floating point number |
| 255 | 0 | infinity |
| 255 | >0 | NaN (not a number) |

(denormalized number: $0.x \times 2^{-126}$)

# Double Precision



- Single precision = 4 bytes
- Double precision = 8 bytes

| Sign | Exponent | Fraction |
|---|---|---|
| 1 bit | 8 bits | 23 bits |
| 1 bit | 11 bits | 52 bits |

# addition

# Addition with Scientific Notation



- Decimal example, with 4 significant digits in encoding
- Example

$$0.1610 + 99.99$$

- In scientific notation

$$1.610 \times 10^{-1} + 9.999 \times 10^{1}$$

- Bring lower number on same exponent as higher number

$$0.01610 \times 10^{1} + 9.999 \times 10^{1}$$

# Addition with Scientific Notation



- Round to 4 significant digits

$$0.016 \times 10^1 + 9.999 \times 10^1$$

- Add fractions

$$0.016 + 9.999 = 10.015$$

- Adjust exponent

$$10.015 \times 10^1 = 1.0015 \times 10^2$$

- Round to 4 significant digits

$$1.002 \times 10^2$$

# Binary Floating Point Addition



- Numbers

$$0.5_{10} = \frac{1}{2}_{10} = \frac{1}{2^1}_{10} = 0.1_2 = 1.000_2 \times 2^{-1}$$

$$-0.4375_{10} = -\frac{7}{16}_{10} = -\frac{7}{2^4}_{10} = 0.0111_2 = -1.110_2 \times 2^{-2}$$

- Bring lower number on same exponent as higher number

$$-1.110 \times 2^{-2} = -0.111 \times 2^{-1}$$

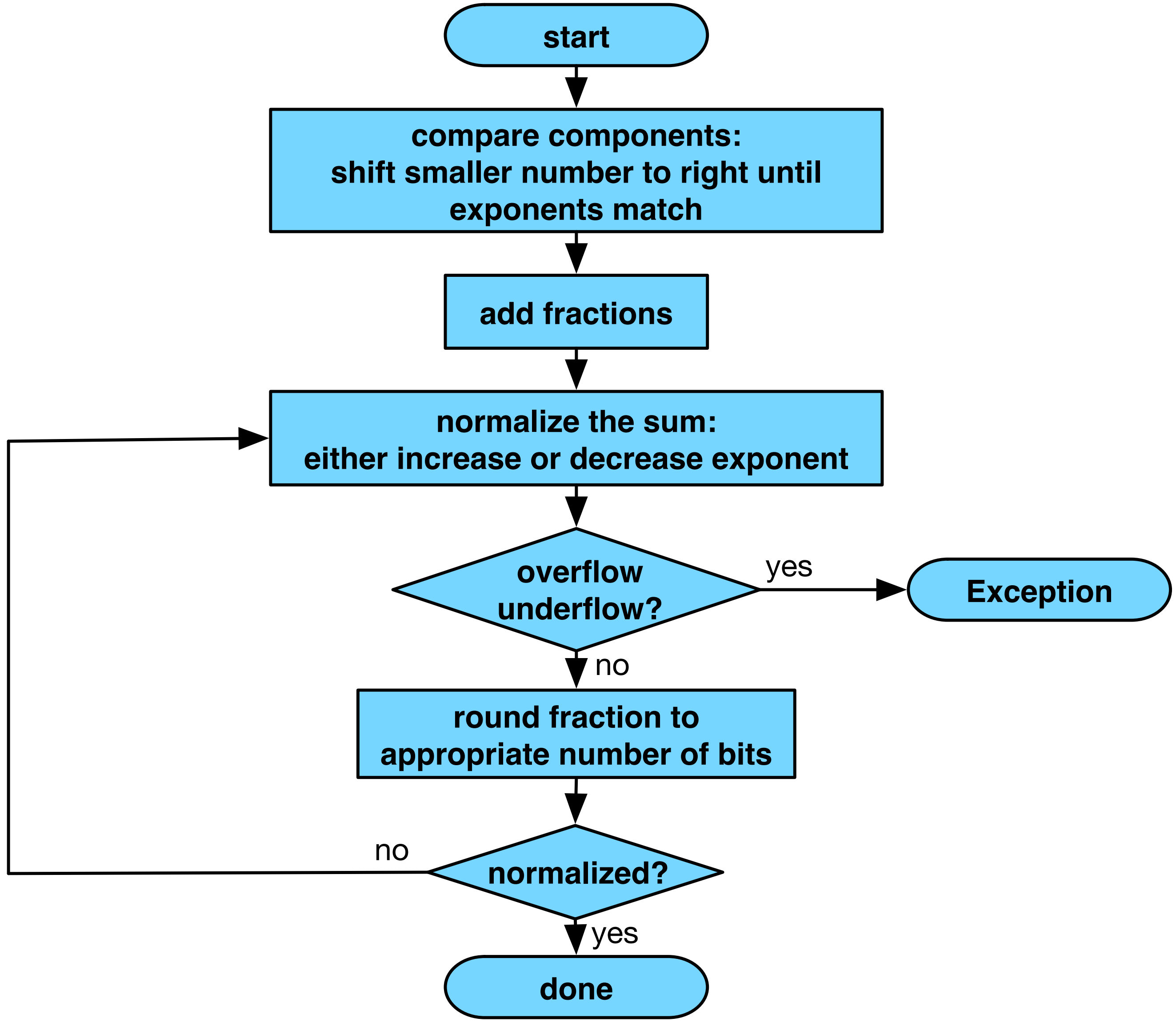
- Add the fractions

$$1.000_2 \times 2^{-1} + (-0.111 \times 2^{-1}) = 0.001 \times 2^{-1}$$

- Adjust exponent

$$0.001 \times 2^{-1} = 1.000 \times 2^{-4}$$

# Flowchart


start
compare components:
shift smaller number to right until
exponents match
add fractions
normalize the sum:
either increase or decrease exponent
overflow
underflow?
yes
Exception
no
round fraction to
appropriate number of bits
no
normalized?
yes
done

# multiplication

# Multiplication with Scientific Notation



- Example: multiply $1.110 \times 10^{10}$ and $9.200 \times 10^{-5}$

$$1.110 \times 10^{10} \times 9.200 \times 10^{-5}$$

$$1.110 \times 9.200 \times 10^{-5} \times 10^{10}$$

$$1.110 \times 9.200 \times 10^{-5+10}$$

- Add exponents

$$-5 + 10 = 5$$

- Multiply fractions

$$1.110 \times 9.200 = 10.212$$

- Adjust exponent

$$10.212 \times 10^{5} = 1.0212 \times 10^{6}$$

# Binary Floating Point Multiplication



- Example

$$1.000 \times 2^{-1} \ \times \ -1.110 \times 2^{-2}$$

- Add exponents

$$-1 + (-2) = -3$$

- Multiply fractions

$$1.000 \times -1.110 = -1.110$$

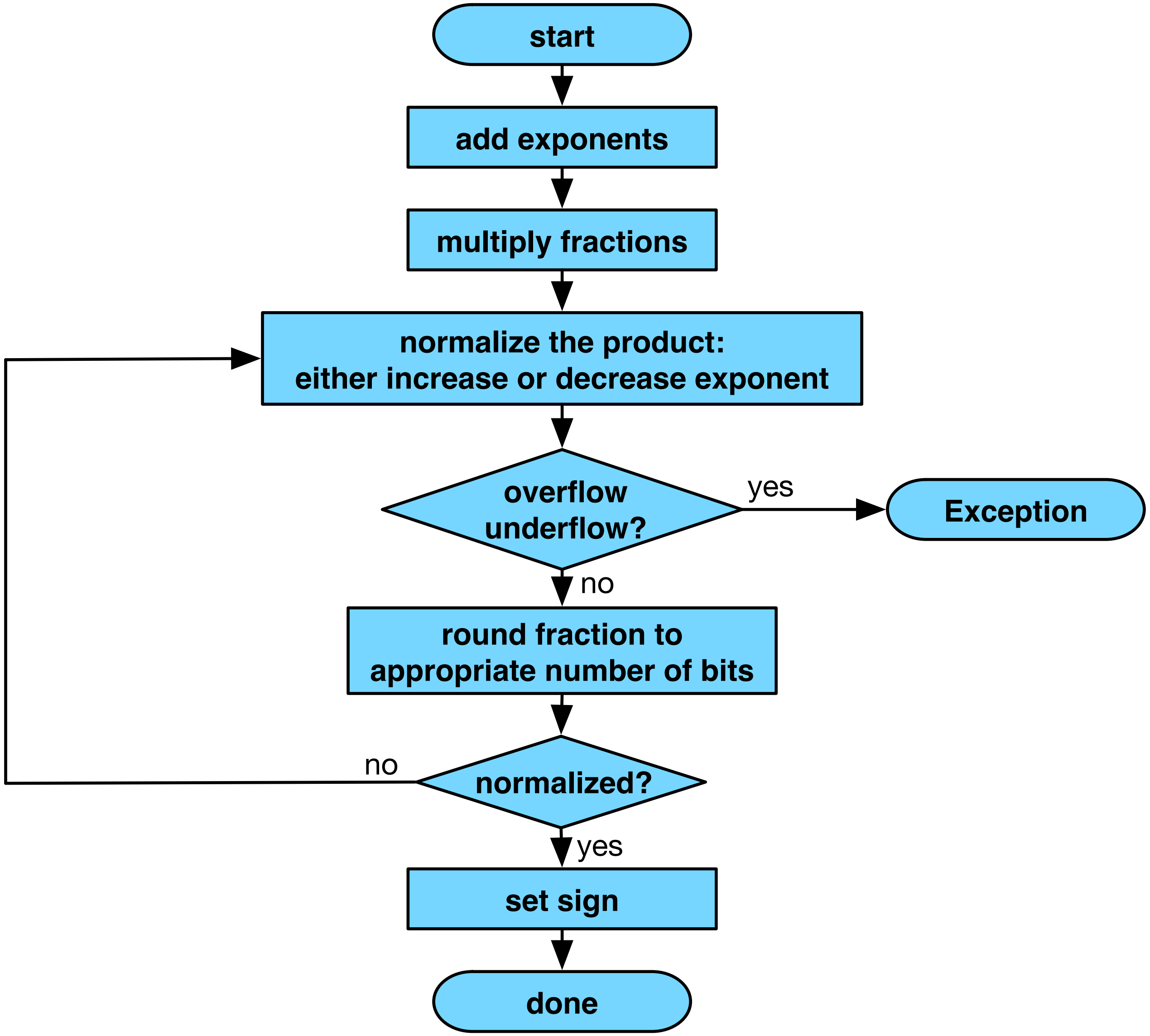
$$1000 \times 1110 = 1110000$$

$$-1.110000$$

- Adjust exponent (not needed)

$$-1.110 \times 2^{-3}$$

# Flowchart


start
add exponents
multiply fractions
normalize the product:
either increase or decrease exponent
overflow
underflow?
yes
Exception
no
round fraction to
appropriate number of bits
no
normalized?
yes
set sign
done

# mips instructions

# Instructions



- Both single precision (s) and double precision (d)
- Addition (add.s / add.d)
- Subtraction (sub.s / sub.d)
- Multiplication (mul.s / mul.d)
- Division (div.s / div.d)
- Comparison (c.x.s / c.x.d)
  - equality (x = eq), inequality (x = neq)
  - less than (x = lt), less than or equal (x = le)
  - greater than (x = gt), greater than or equal (x = ge)
- Floating point branch on true (bclt) or fals (bclf)

# Floating Point Registers



- MIPS has a separate set of registers for floating point numbers
- Little overhead, since used for different instructions
  - no need to specify in add, subtract, etc. instruction codes
  - different wiring for floating point / integer registers
  - much more limited use for floating point registers (e.g., never an address)
- Double precision = 2 registers used

# Example



- Conversion Fahrenheit to Celsius (5.0/9.0 $\times$ (x - 32.0))
- Input value x stored in register $f12, constant in offsets to $gp
- Code

```
lwcl    $f16, const5($gp)     ; load 5.0
lwcl    $f18, const9($gp)     ; load 9.0
div.s   $f16, $f16, $f18      ; $f16 = 5.0/9.0
lwcl    $f18, const32($gp)    ; load 32.0
sub.s   $f18, $f12, $f18      ; $f18 = x-32.0
mul.s   $f0, $f16, $f18       ; $f0 = result
```