

# Lecture 5: Floating point

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# Floating point numbers

# Numbers

- ▶ So far, we only dealt with integers
- ▶ But there are other types of numbers

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- ▶ Rational numbers (from ratio  $\simeq$  fraction)
  - ▶  $3/4 = 0.75$
  - ▶  $10/3 = 3.33333333\dots$

# Numbers

- ▶ So far, we only dealt with integers
- ▶ But there are other types of numbers
- ▶ Rational numbers (from ratio  $\simeq$  fraction)
  - ▶  $3/4 = 0.75$
  - ▶  $10/3 = 3.33333333\dots$
- ▶ Real numbers
  - ▶  $\pi = 3.14159265\dots$
  - ▶  $e = 2.71828182\dots$

# Very Large Numbers

- ▶ Distance of sun and earth

150,000,000,000 meters

- ▶ Scientific notation

$1.5 \times 10^{11}$  meters

- ▶ Another example: number of atoms in 12 gram of carbon-12 (1 mol)

$6.022140857 \times 10^{23}$

# Binary Numbers in Scientific Notation

- ▶ Example binary number ( $\pi$  again)

11.0010010001

- ▶ Scientific notation

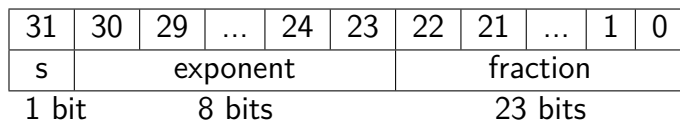
$1.10010010001 \times 2^1$

- ▶ General form

$1.x \times 2^y$

# Representation

- ▶ IEEE 754 floating point standard
- ▶ Uses 4 bytes



- ▶ Exponent is offset with a bias of 127  
e.g.  $2^{-6} \rightarrow \text{exponent} = -6 + 127 = 121$



# Conversion into Binary

- ▶  $\pi = 3.14159265$
- ▶ Number before period:  $3_{10} = 11_2$
- ▶ Conversion of fraction .14159265

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	$0.14159265 \times 2 \downarrow$
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	$0.14159265 \times 2 \downarrow$
0	$0.2831853 \times 2 \downarrow$
0	$0.5663706 \times 2 \downarrow$
1	$0.1327412$

# Conversion into Binary

- ▶  $\pi = 3.14159265$
- ▶ Number before period:  $3_{10} = 11_2$
- ▶ Conversion of fraction .14159265

Digit	Calculation
	$0.14159265 \times 2 \downarrow$
0	$0.2831853 \times 2 \downarrow$
0	$0.5663706 \times 2 \downarrow$
1	$0.1327412 \times 2 \downarrow$
0	$0.2654824 \times 2 \downarrow$
0	$0.5309648 \times 2 \downarrow$
1	$0.0619296 \times 2 \downarrow$
0	$0.1238592 \times 2 \downarrow$
0	$0.2477184 \times 2 \downarrow$
0	$0.4954368 \times 2 \downarrow$
0	$0.9908736 \times 2 \rightarrow$

Digit	Calculation
1	$0.9817472 \times 2 \downarrow$
1	$0.9634944 \times 2 \downarrow$
1	$0.9269888 \times 2 \downarrow$
1	$0.8539776 \times 2 \downarrow$
1	$0.7079552 \times 2 \downarrow$
1	$0.4159104 \times 2 \downarrow$
0	$0.8318208 \times 2 \downarrow$
1	$0.6636416 \times 2 \downarrow$
1	$0.3272832 \times 2 \downarrow$
0	$0.6545664 \times 2 \downarrow$
1	$0.3091328 \times 2$

- ▶ Binary: 11.00100100001111101101

# Encoding into Representation

- ▶  $\pi$

$$1.1001001000011111101101 \times 2^1$$

- ▶ Encoding

Sign	Exponent	Fraction
0	10000000	1001001000011111101101

- ▶ Note: leading 1 in fraction is omitted

# Special Cases

► Zero



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- ▶ Negative infinity ( $-1/0$ )
- ▶ Not a number ( $0/0$  or  $\infty - \infty$ )

Exponent	Fraction	Object
0	0	zero
0	>0	denormalized number
1-254	anything	floating point number
255	0	infinity
255	>0	NaN (not a number)

(denormalized number:  $0.x \times 2^{-126}$ )

# Clicker quiz!

Clicker quiz omitted from public slides

# Double Precision

- ▶ Single precision = 4 bytes

<b>Sign</b>	<b>Exponent</b>	<b>Fraction</b>
1 bit	8 bits	23 bits

- ▶ Double precision = 8 bytes

<b>Sign</b>	<b>Exponent</b>	<b>Fraction</b>
1 bit	11 bits	52 bits

# Addition

# Addition with Scientific Notation

- ▶ Decimal example, with 4 significant digits in encoding
- ▶ Example

$$0.1610 + 99.99$$

- ▶ In scientific notation

$$1.610 \times 10^{-1} + 9.999 \times 10^1$$

# Addition with Scientific Notation

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- ▶ Example

$$0.1610 + 99.99$$

- ▶ In scientific notation

$$1.610 \times 10^{-1} + 9.999 \times 10^1$$

- ▶ Bring lower number on same exponent as higher number

$$0.01610 \times 10^1 + 9.999 \times 10^1$$



# Addition with Scientific Notation

- ▶ Round to 4 significant digits

$$0.016 \times 10^1 + 9.999 \times 10^1$$

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$$0.016 + 9.999 = 10.015$$

- ▶ Adjust exponent

$$10.015 \times 10^1 = 1.0015 \times 10^2$$

# Addition with Scientific Notation

- ▶ Round to 4 significant digits

$$0.016 \times 10^1 + 9.999 \times 10^1$$

- ▶ Add fractions

$$0.016 + 9.999 = 10.015$$

- ▶ Adjust exponent

$$10.015 \times 10^1 = 1.0015 \times 10^2$$

- ▶ Round to 4 significant digits

$$1.002 \times 10^2$$

# Binary Floating Point Addition

► Numbers

$$0.5_{10} = \frac{1}{2}_{10}$$

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$$0.5_{10} = \frac{1}{2}_{10} = \frac{1}{2^1}_{10} = 0.1_2 = 1.000_2 \times 2^{-1}$$

$$-0.4375_{10} = -\frac{7}{16}_{10}$$

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- ▶ Bring lower number on same exponent as higher number

$$-1.110 \times 2^{-2} = -0.111 \times 2^{-1}$$

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- ▶ Bring lower number on same exponent as higher number

$$-1.110 \times 2^{-2} = -0.111 \times 2^{-1}$$

- ▶ Add the fractions

$$1.000_2 \times 2^{-1} + (-0.111 \times 2^{-1}) = 0.001 \times 2^{-1}$$

# Binary Floating Point Addition

- ▶ Numbers

$$0.5_{10} = \frac{1}{2}_{10} = \frac{1}{2^1}_{10} = 0.1_2 = 1.000_2 \times 2^{-1}$$

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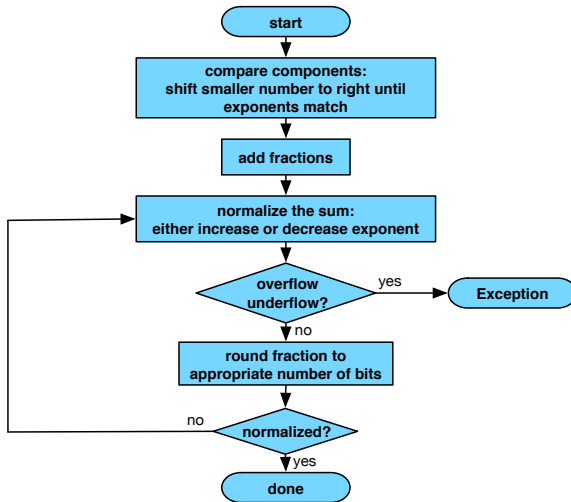
- ▶ Add the fractions

$$1.000_2 \times 2^{-1} + (-0.111 \times 2^{-1}) = 0.001 \times 2^{-1}$$

- ▶ Adjust exponent

$$0.001 \times 2^{-1} = 1.000 \times 2^{-4}$$

# Flowchart



# Multiplication



# Multiplication with Scientific Notation

- ▶ Example: multiply  $1.110 \times 10^{10}$  and  $9.200 \times 10^{-5}$

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$$1.110 \times 9.200 \times 10^{-5} \times 10^{10}$$

$$1.110 \times 9.200 \times 10^{-5+10}$$

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$$1.110 \times 9.200 \times 10^{-5} \times 10^{10}$$

$$1.110 \times 9.200 \times 10^{-5+10}$$

- ▶ Add exponents

$$-5 + 10 = 5$$

# Multiplication with Scientific Notation

- ▶ Example: multiply  $1.110 \times 10^{10}$  and  $9.200 \times 10^{-5}$

$$1.110 \times 10^{10} \times 9.200 \times 10^{-5}$$

$$1.110 \times 9.200 \times 10^{-5} \times 10^{10}$$

$$1.110 \times 9.200 \times 10^{-5+10}$$

- ▶ Add exponents

$$-5 + 10 = 5$$

- ▶ Multiply fractions

$$1.110 \times 9.200 = 10.212$$

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- ▶ Example: multiply  $1.110 \times 10^{10}$  and  $9.200 \times 10^{-5}$

$$1.110 \times 10^{10} \times 9.200 \times 10^{-5}$$

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$$1.110 \times 9.200 \times 10^{-5+10}$$

- ▶ Add exponents

$$-5 + 10 = 5$$

- ▶ Multiply fractions

$$1.110 \times 9.200 = 10.212$$

- ▶ Adjust exponent

$$10.212 \times 10^5 = 1.0212 \times 10^6$$

# Binary Floating Point Multiplication

► Example

$$1.000 \times 2^{-1} \times -1.110 \times 2^{-2}$$



# Binary Floating Point Multiplication

- ▶ Example

$$1.000 \times 2^{-1} \times -1.110 \times 2^{-2}$$

- ▶ Add exponents

$$-1 + (-2) = -3$$

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$$1.000 \times 2^{-1} \times -1.110 \times 2^{-2}$$

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- ▶ Multiply fractions

$$1.000 \times -1.110 = -1.110$$

# Binary Floating Point Multiplication

- ▶ Example

$$1.000 \times 2^{-1} \times -1.110 \times 2^{-2}$$

- ▶ Add exponents

$$-1 + (-2) = -3$$

- ▶ Multiply fractions

$$1.000 \times -1.110 = -1.110$$

$$1000 \times 1110 = 1110000$$

# Binary Floating Point Multiplication

- ▶ Example

$$1.000 \times 2^{-1} \times -1.110 \times 2^{-2}$$

- ▶ Add exponents

$$-1 + (-2) = -3$$

- ▶ Multiply fractions

$$1.000 \times -1.110 = -1.110$$

$$1000 \times 1110 = 1110000$$

$$-1.110000$$

# Binary Floating Point Multiplication

- ▶ Example

$$1.000 \times 2^{-1} \times -1.110 \times 2^{-2}$$

- ▶ Add exponents

$$-1 + (-2) = -3$$

- ▶ Multiply fractions

$$1.000 \times -1.110 = -1.110$$

$$1000 \times 1110 = 1110000$$

$$-1.110000$$

- ▶ Adjust exponent (not needed)

$$-1.110 \times 2^{-3}$$

# Flowchart

